Geometry of canonical Kähler metrics

on ~~Kähler manifolds~~

algebraic variety

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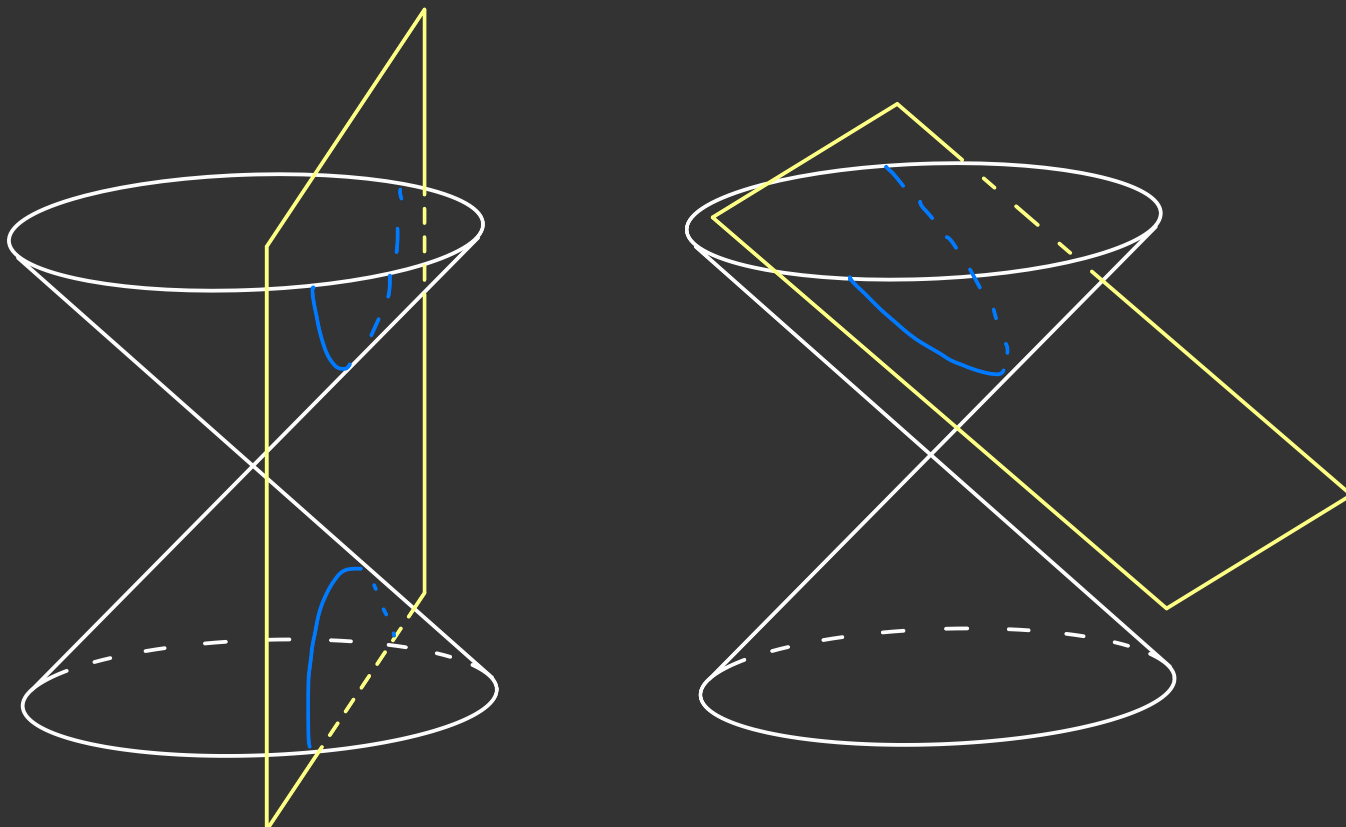
Contents

§ 1. Algebraic variety

Conic curve (Apollonius, around BC 200)



hyperbolic　　parabolic　　elliptic

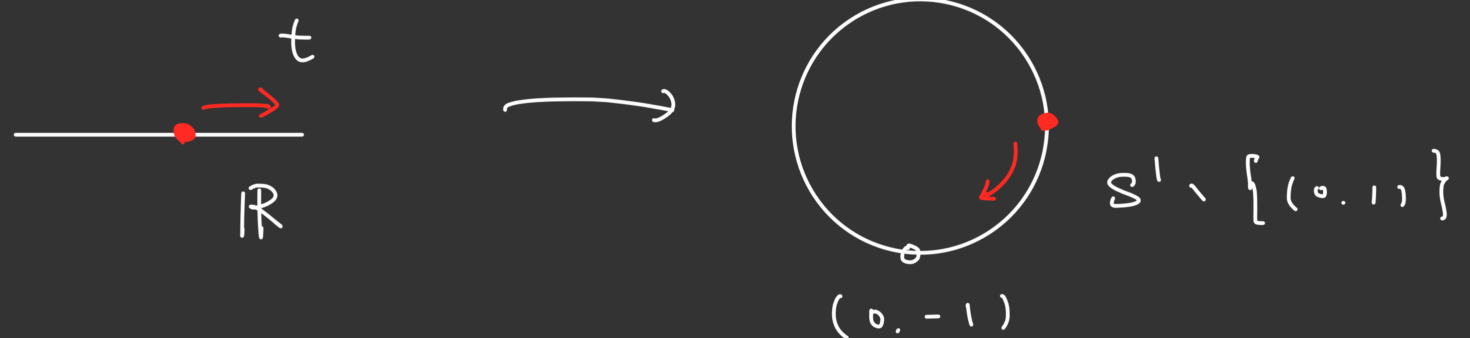
$$D := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0 \}$$

$H \subset \mathbb{R}^3$ plane　　$C_H = D \cap H$

Can we <u>identify</u> these curves?

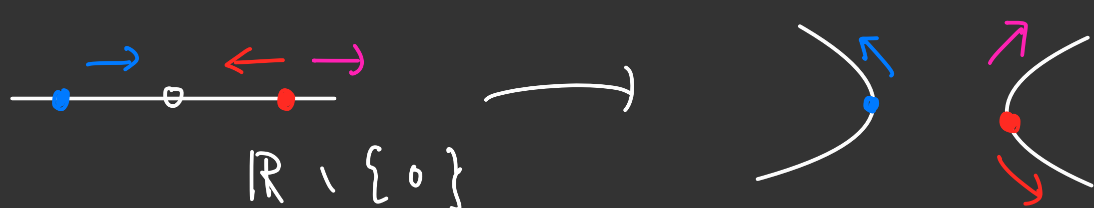
<u>Rational parametrization</u>

- $x^2 + y^2 = 1 \ : \ (x, y) = \left(\dfrac{2t}{1+t^2}, \dfrac{1-t^2}{1+t^2} \right)$



- $y = x^2 \ : \ (x, y) = (t, t^2)$

- $x^2 - y^2 = 1 \ : \ (x, y) = \left(\dfrac{t^2+1}{2t}, \dfrac{t^2-1}{2t} \right)$



Compactification

$$\mathbb{R}P^n = \{ (X_0 : X_1 : \cdots : X_n) \mid (X_0, \cdots, X_n) \in \mathbb{R}^{n+1} \setminus 0 \}$$

$$(X_0 : X_1 : \cdots : X_n) = (Y_0 : Y_1 : \cdots : Y_n)$$

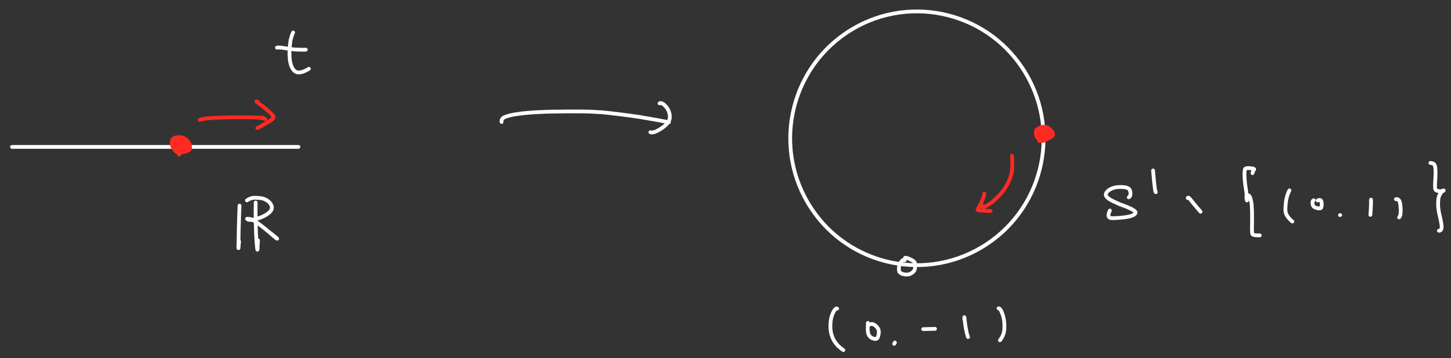
$$\text{iff} \quad \exists \lambda \in \mathbb{R} \setminus 0 \quad \text{s.t.} \quad \forall i \quad X_i = \lambda Y_i$$

$$\varphi_i : \mathbb{R}^n \hookrightarrow \mathbb{R}P^n$$

$$\begin{array}{ccc} & & \begin{array}{ccccc} 0 & \cdots & i & \cdots & n \end{array} \\ (x_1, \cdots, x_n) & \longmapsto & (x_1 : \cdots : 1 : \cdots : x_n) \end{array}$$

for $C \subset \mathbb{R}^2$, we put

$$\overline{C} := \{ \lim_{i \to \infty} (1 : x_i : y_i) \in \mathbb{R}P^2 \mid \{(x_i, y_i)\} \subset C \}$$



$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d, hook, "\varphi_0"'] & \mathbb{R}^2 \\ \mathbb{R}P^1 \arrow[ur] & \end{tikzcd}$$

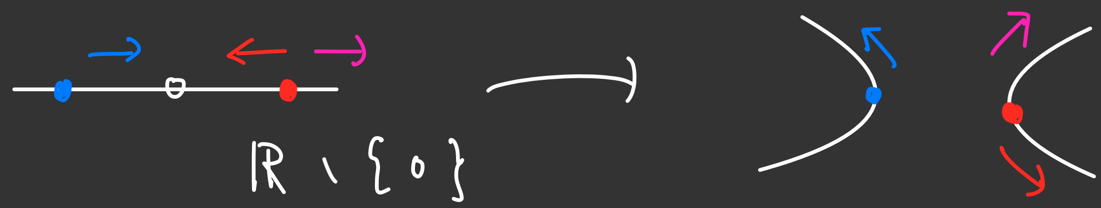
$$t \longmapsto \left(\frac{2t}{1+t^2} , \frac{1-t^2}{1+t^2} \right)$$

$$(S:T) \longmapsto \left(\frac{2ST}{S^2+T^2} , \frac{S^2-T^2}{S^2+T^2} \right)$$

$$(0:1) \longmapsto (0,-1)$$

$$\parallel$$

$$\lim_{t \to \pm\infty} (1:t)$$



$$\begin{tikzcd} \mathbb{R}\setminus\{0\} \arrow[r] \arrow[d, hook, "\varphi_0"'] & \mathbb{R}^2 \arrow[d, hook] \\ \mathbb{R}P^1 \arrow[r] & \mathbb{R}P^2 \end{tikzcd}$$

$$\mathbb{R}\setminus\{0\} \to \mathbb{R}^2 : t \mapsto \left(\frac{t^2+1}{2t} , \frac{t^2-1}{2t} \right)$$

$$\mathbb{R}P^1 \to \mathbb{R}P^2 : (S:T) \mapsto (2ST : T^2+S^2 : T^2-S^2)$$

$$\mathbb{R}P^1 = S^1$$

$$(1:0) \mapsto (0:1:-1)$$

$$(1:0) = \lim_{t \to \pm 0} (1\ t)$$

$$(0:1) \mapsto (0:1:1)$$

$$(0:1) = \lim_{t \to \pm\infty} (1:t)$$

$$\boxed{\begin{array}{ccc} \mathbb{R}P^1 & \xrightarrow{\sim} & \overline{\mathbb{C}}_H \\ \shortparallel & & \\ S^1 & & \end{array}}$$

We often assume compactness for finiteness or well-definedness of various invariants & notions (cohomology, intersection number, ...)

We would like to study higher dimensional analogue.

Before going further, let's observe lower dimensional analogue.

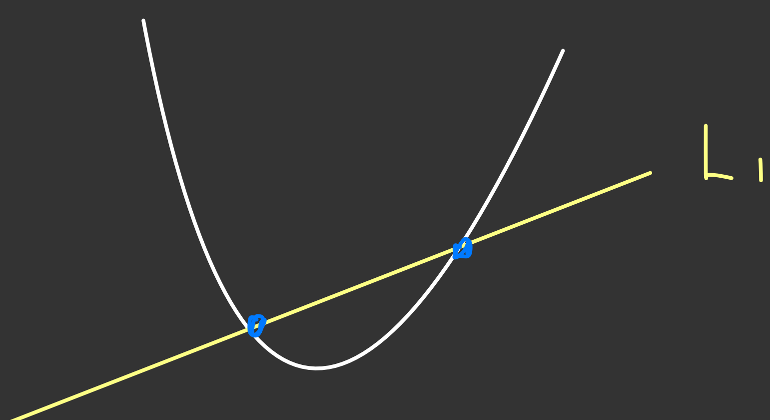
Replace $\mathbb{R}$ with $\mathbb{C}$

$$C = \{ (x, y) \in \cancel{\mathbb{R}^2}\ \mathbb{C}^2 \mid y - x^2 = 0 \}$$

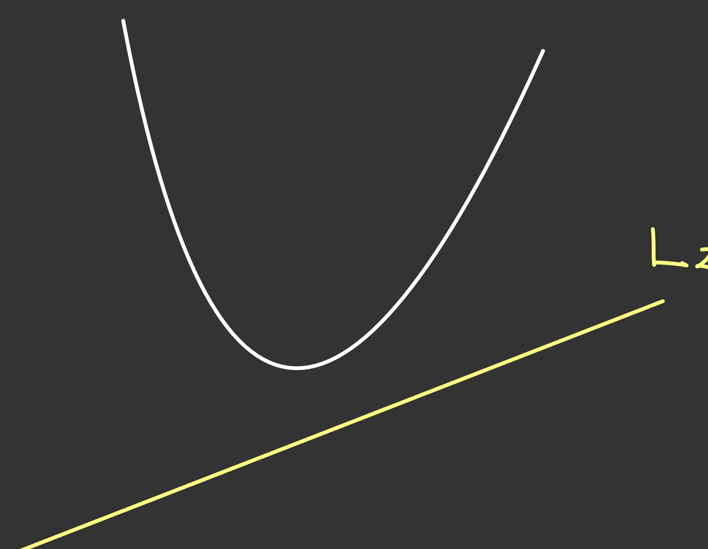
$L \subset \cancel{\mathbb{R}^2}\ \mathbb{C}^2$ line.

$$P_L := C \cap L$$

(No intersection at ∞)



$$P_{L_1} \not\simeq P_{L_2}$$



<u>Thm</u> (fundamental theorem of algebra)

degree d complex valued polynomial
has d-roots in complex number

$\rightsquigarrow$ Geometry of the solution set
of complex valued (multi-variable)
polynomials
seems well-described by
the property of the polynomials.

Algebraic variety / $\mathbb{C}$

$$\mathbb{C}P^n = \{ (Z_0 : Z_1 : \cdots : Z_n) \mid (Z_0, \cdots, Z_n) \in \mathbb{C}^{n+1} \setminus 0 \}$$

$$(Z_0 : Z_1 : \cdots : Z_n) = (W_0 : \cdots : W_n)$$

iff $\exists \lambda \in \mathbb{C} \setminus 0$ s.t. $\forall i \quad Z_i = \lambda W_i$

$$\varphi_i : \mathbb{C}^n \to \mathbb{C}P^n$$

$$(Z_1, \cdots, Z_n) \longmapsto (\overset{0}{Z_1} : \cdots : \overset{i}{1} : \cdots : \overset{n}{Z_n})$$

(projective) algebraic variety $/\mathbb{C}$

is expressed as

$$V = \{ (z_0 : \cdots : z_n) \in \mathbb{C}P^n \mid p_1(z_0, \cdots, z_n) = \cdots = p_k(z_0, \cdots, z_n) = 0 \}$$

p_j are homogeneous polynomials

We are interested in its intrinsic geometry

holomorphic structure

e.g. (quadratic curve) $Q = \{ (z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid a_0 z_0^2 + a_1 z_1^2 + a_2 z_2^2 + b_0 z_1 z_2 + b_1 z_0 z_2 + b_2 z_0 z_1 = 0 \}$

quadratic form can be diagonalized

$\rightsquigarrow$ ${}^\exists A \in GL(3, \mathbb{C})$

s.t. $\varphi_A(Q) = Q_0 = \{ (z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_0^2 + z_1^2 - z_2^2 = 0 \}$

where $\varphi_A : \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$

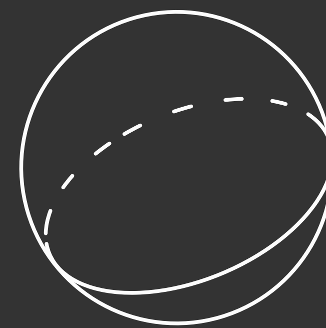
$(z_0 : z_1 : z_2) \longmapsto (z_0 : z_1 : z_2) A$

$$\mathbb{CP}^1 \to \mathbb{CP}^2 : (z_0 : z_1) \longmapsto (2z_0 z_1 : z_1^2 - z_2^2 : z_1^2 + z_2^2)$$

gives the following

$$\mathbb{CP}^1 \xrightarrow{\sim} Q_0 \simeq Q$$

biholomorphic



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diffeomorphic

$$S^2 \xrightarrow{\sim} \mathbb{CP}^1$$

$$(x, y, z) \longmapsto (x + \sqrt{-1}\,y : 1 - z)$$

$$= (1 + z : x - \sqrt{-1}\,y)$$

$$\left((1+z)(1-z) = 1 - z^2 = x^2 + y^2 = (x + \sqrt{-1}\,y)(x - \sqrt{-1}\,y) \right)$$

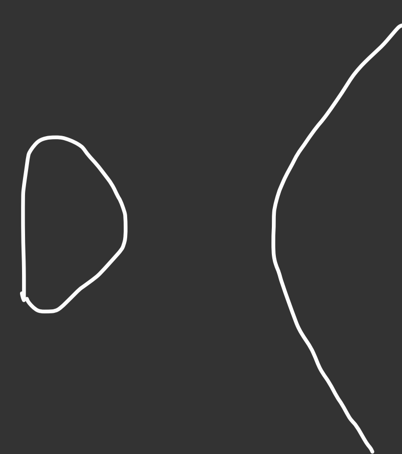
e.g. (elliptic curve)

$$E = \{ (Z_0 : Z_1 : Z_2) \in \mathbb{C}P^2 \mid Z_2^2 Z_0 - 4(Z_1 - e_1 Z_0)(Z_1 - e_2 Z_0)(Z_1 - e_3 Z_0) = 0 \}$$

$$\varphi_0^{-1}(E) = \{ (x, y) \in \mathbb{C}^2 \mid y^2 = 4(x - e_1)(x - e_2)(x - e_3) \}$$

Images of $\varphi_0^{-1}(E) \cap \mathbb{R}^2$

$y^2 = x^3 - x + 1$

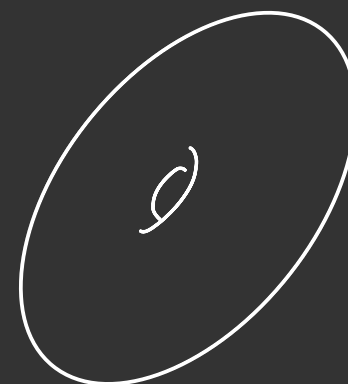


$y^2 = x^3 - x$

Weierstrass $\wp$ function (1872)

$$\Lambda_\tau = \{ n + \tau m \in \mathbb{C} \mid n, m \in \mathbb{Z} \}, \quad \operatorname{Im} \tau > 0$$

$$\mathbb{C} / \Lambda_\tau = \{ [z] \mid z \in \mathbb{C} \}$$

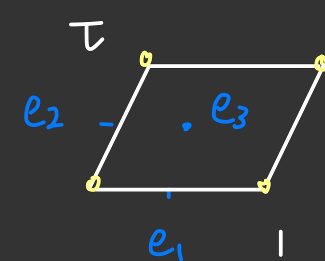


$$[z] = [w]$$

iff $\exists\, n, m \in \mathbb{Z}$ s.t. $z = w + n + \tau m$

$$\wp : \mathbb{C} \setminus \Lambda_\tau \xrightarrow{\text{holomophic}} \mathbb{C}$$

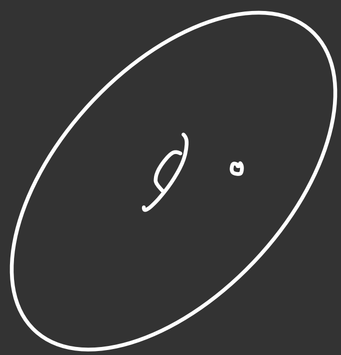
$$z \longmapsto \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right)$$



$$\left(\begin{array}{l} \cdot\ \wp(z + n + \tau m) = \wp(z) \\ \cdot\ (\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \end{array} \right.$$

$$\rightsquigarrow \quad f : (\mathbb{C} \setminus \Lambda_\tau)/\Lambda_\tau \longrightarrow \mathbb{CP}^2$$

$$[z] \longmapsto (1 : \wp(z) : \wp'(z))$$



It extends to $\mathbb{C}/\Lambda_\tau \xrightarrow{\sim} E_{e_1, e_2, e_3}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, dashed, "\exists \varphi"] \arrow[d, "\text{universal cover}"'] & \mathbb{C} \arrow[d] \\
\mathbb{C}/\Lambda_\tau \arrow[r, dashed, "\exists \bar{\varphi}"] & \mathbb{C}/\Lambda_{\tau'}
\end{tikzcd}$$

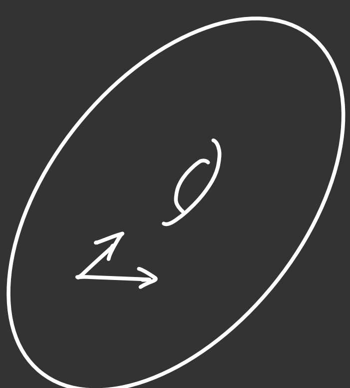
$\varphi(\Lambda_\tau) = \Lambda_{\tau'}$

iff

τ & τ' are modular

i.e. $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$



holomorphic structure
||
conformal structure

$\mathbb{H} / SL(2,\mathbb{Z})$: <u>moduli</u> of elliptic curves.

$$\begin{array}{c} \mathbb{H}/SL(2,\mathbb{Z}) \\ \text{j-invariant} \downarrow \wr \\ \mathbb{C} \end{array}$$

$\mathbb{H} \circlearrowleft SL(2,\mathbb{Z})$ is closed but not free

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}$$

$$g_2(\tau) = 60 \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-4}$$

$$g_3(\tau) = 140 \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-6}$$

Riemann's uniformization theorem

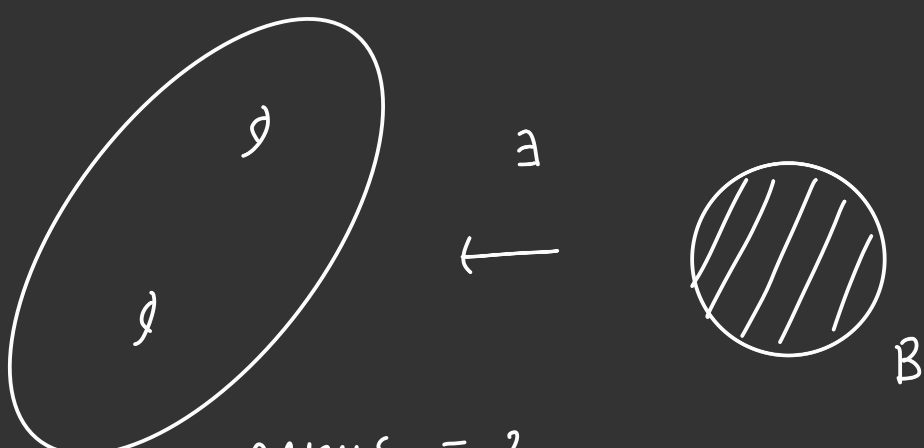
Every 1-dimensional smooth algebraic variety

is a quotient space of one of

$$\mathbb{CP}^1 , \quad \mathbb{C} , \quad B = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

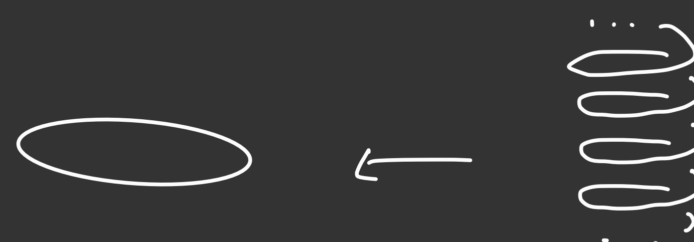
not biholomorphic

(Liouville theorem)



genus = 2

image of universal cover



$$S^1 \longleftarrow \mathbb{R}$$

$$e^{i\theta} \longleftarrow\!\!\!\shortmid \ \theta$$

§ 2. Kähler-Einstein metric

Recall algebraic variety is written as

$$V = \{ (z_0 : \cdots : z_n) \in \mathbb{C}P^n \mid p_1(z_0, \cdots, z_n) = \cdots = p_k(z_0, \cdots z_n) = 0 \}$$

p_j are homogeneous polynomials

Smooth variety

for each $i = 0, 1, \cdots, n$, $j = 1, \cdots, k$

$p_j \circ \varphi_i : \mathbb{C}^n \to \mathbb{C}$ is a polynomial

$$S_i := (p_1 \circ \varphi_i, \cdots, p_k \circ \varphi_i) : \mathbb{C}^n \to \mathbb{C}^k$$

Suppose $k \le n$ and Jacobian $D_x s_i$ has rank k at $x \in \varphi_i^{-1}(V)$.

$\rightsquigarrow$ $B = \{ z \in \mathbb{C}^{n-k} \mid |z| < 1 \}$

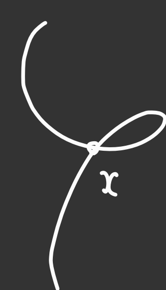
implicit function theorem

$$\begin{array}{ccc} & B & \\ \exists\,\psi & \downarrow & \text{holomorphic} \\ & \varphi_i^{-1}(V) & \subset \mathbb{C}^n \\ & \downarrow & \downarrow \varphi_i \\ & V & \subset \mathbb{CP}^n \end{array}$$

s.t.
- $\psi(0) = x$
- ψ : injective
- $\mathrm{rk}\, D_b \psi = n-k$

$\rightsquigarrow$ <u>chart</u> of V at x.

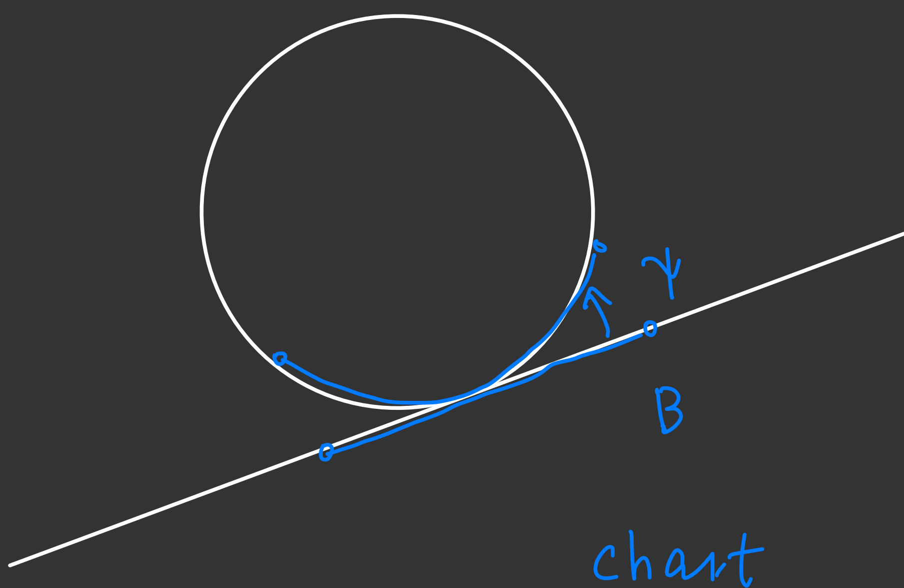
V is smooth at x

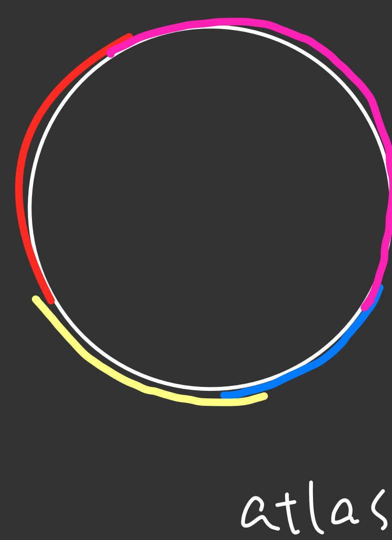


singular at x

$\rightsquigarrow$ <u>atlas</u> of V whose coordinate changes are holomorphic

Note. even when V is given by explicit equations, charts $\gamma : B \to V$ may be quite implicit.





Kähler metric

A Kähler metric on $B = \{ z \in \mathbb{C}^{\ell} \mid |z| < 1 \}$

is a smooth map $g : B \to \mathrm{Herm}^{+}(\ell \times \ell)$

s.t. $\exists f : B \to \mathbb{R}$ Kähler potential

$$g(z) = \left(\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}(z) \right)_{i,j}$$ complex Hessian

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \frac{1}{\sqrt{-1}} \frac{\partial}{\partial y^i} \right)$$

$$\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \frac{1}{\sqrt{-1}} \frac{\partial}{\partial y^j} \right)$$

The <u>Ricci curvature</u> of a Kähler metric g on B is the following map

$$\mathrm{Ric}(g) : B \to \mathrm{Herm}(\ell \times \ell)$$

$$\mathrm{Ric}(g)(z) := -\left(\frac{\partial^2 \log \det g}{\partial z^i \, \partial \bar{z}^j}(z) \right)_{i,j}$$

<u>Rem</u> This coincides with the usual Ricci curvature in Riemannian geometry. (for a Kähler metric g)

e.g.

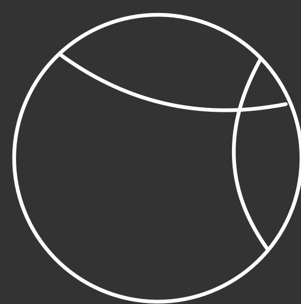
- Euclidean metric (flat metric)

$$g_E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad \text{constant fct}$$

$$f_E = |z|^2. \qquad Ric(g_E) = 0$$

- Poincare metric (hyperbolic metric)

$$g_P = 2\frac{1}{(1-|z|^2)^2}\begin{pmatrix} 1-|z|^2+|z_1|^2 & & \bar{z}_i z_j \\ & \ddots & \\ \bar{z}_i z_j & & 1-|z|^2+|z_\ell|^2 \end{pmatrix}$$



$$f_P = -2\log(1-|z|^2). \qquad Ric(g_P) = -g_P$$

a Fubini - Study metric

$$g_{FS} = 2 \frac{1}{(1+|z|^2)^2} \begin{pmatrix} 1+|z|^2-|z_1|^2 & & -\bar{z}_i z_j \\ & \ddots & \\ -\bar{z}_i z_j & & 1+|z|^2-|z_\ell|^2 \end{pmatrix}$$

$$f_{FS} = 2 \log (1+|z|^2) .$$ $Ric(g_{FS}) = g_{FS}$

$\rightsquigarrow$ well-defined for $z \in \mathbb{C}^\ell$

$\rightsquigarrow$ We can extend this metric to a metric on $\mathbb{CP}^\ell$

(g_E cannot be extended to $\mathbb{CP}^\ell$)

Kähler metric on algebraic variety

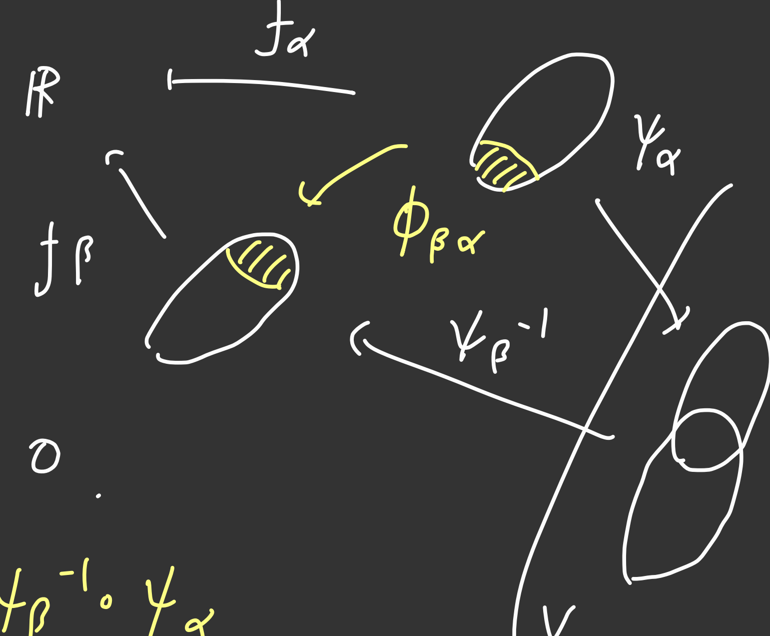
V : a smooth variety

$\{\psi_\alpha : B \to V\}_{\alpha \in A}$: an atlas of V

A <u>Kähler metric on V</u> is a collection

$\{g_\alpha : B \to \mathrm{Herm}^+\}_{\alpha \in A}$ of Kähler metrics on

B. s.t. $\forall\, \alpha, \beta \in A$



$$\frac{\partial^2 (f_\alpha - f_\beta \circ \phi_{\beta\alpha})}{\partial z^i \, \partial \bar{z}^j} = 0.$$

$\phi_{\beta\alpha} = \psi_\beta^{-1} \circ \psi_\alpha$

<u>Remark</u> This is equivalent to give a Riemannian metric tensor g

s.t. $\begin{cases} 1. & g(J-, J-) = g \\ 2. & \text{For } \omega, \quad \omega(-,-) = g(J-,-) \\ & d\omega = 0 \end{cases}$

where J : $J \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$

$$J \frac{\partial}{\partial y^i} = - \frac{\partial}{\partial x^i}$$

$\rightsquigarrow$ Restriction of a Kähler metric to any complex submanifold is Kähler

$\rightsquigarrow$ alg variety $V \subset \mathbb{C}P^n$ has a Kähler met.

Kodaira (小平邦彦) uses the existence of Kähler metric to show various properties of algebraic variety

(cf. Hodge decomposition

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$
)

Description of metric is unimportant

Kähler-Einstein metric

A Kähler metric $g = \{g_\alpha\}_{\alpha \in A}$ on V

is called Kähler-Einstein

if $\exists \lambda \in \mathbb{R}$ s.t. $\forall \alpha \in A$

$$Ric(g_\alpha) = \lambda g_\alpha$$

e.g. 1. $(\mathbb{C}^n, g_E)$, (B^n, g_P), $(\mathbb{CP}^n, g_{FS})$

are KE

2. Any 1-dim smooth variety admits KE.

<u>Thm</u> (Miyaoka - Yau)

If V admits a Kähler-Einstein metric,

then $\left(c_1(V)^2 - \frac{2(n+1)}{n} c_2(V) \quad L^{n-2}\right) \leq 0$

with the equality iff the universal cover

of V is biholomorphic to one of

$$\mathbb{C}P^n, \quad \mathbb{C}^n, \quad B^n$$

$$L = [\, g_\alpha\, ij\, dz^i \wedge d\bar{z}^j \,] \in H^2(X, \mathbb{R})$$

$$c_1(V) = [\, \mathrm{Ric}(g_\alpha)_{ij}\, dz^i \wedge d\bar{z}^j \,] \in H^2(X, \mathbb{R})$$

Classical theorems on KE met

Thm (Calabi, Bando-Mabuchi, Berman-Berndtsson)

Given a smooth variety V,

KE metrics are unique in a suitable sense.

if it exists.

$\rightsquigarrow$ Riemannian geometry of KE metric

must reflect algebraic geometry of V.

If V admits KE met $\mathrm{Ric}(g) = \lambda g$.

then $c_1(V) = \lambda L$.

<u>Thm</u> (Aubin Yau)

Suppose V satisfies $c_1(V) = \lambda L$ for $\lambda \leq 0$,

then V admits a KE met (in L).

<u>Rem</u>

If $\lambda = 0$, then $c_1(V) = 0$.

Such variety is called Calabi-Yau variety.

CY variety $\nexists$ $\exists$ Ricci flat KE e.g.

degree d polynomial

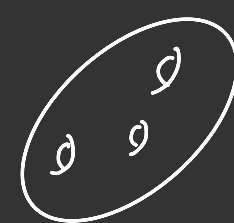
Suppose $V = \{ z \in \mathbb{C}P^{n+1} \mid p(z) = 0 \}$.

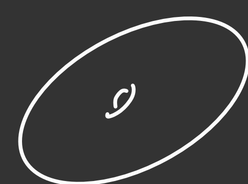
$d > n+2 \Rightarrow c_1(V) < 0 \quad \exists$ KE

$d = n+2 \Rightarrow c_1(V) = 0 \quad \exists \, Ric = 0$

$d < n+2 \quad$ partial results (2021)

When $n = 1$.

$d > 3 \quad$ genus $\frac{1}{2}(d-1)(d-2)$ · curve 

$d = 3 \quad$ elliptic curve 

$d < 3 \quad \mathbb{C}P^1$ 

Fano variety

When $c_1(V) > 0$,

V may not admit KE metric

e.g. $\mathbb{C}P^2$ FS

$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ $\nexists$ KE

$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ $\exists$ KE

Kähler class

Given a Kähler metric $g = \{g_\alpha\}_{\alpha \in A}$ on V,

we can perturb it

by a global function $f : V \to \mathbb{R}$

as

$$g' := \left\{ g_\alpha + \left(\frac{\partial^2 f \circ \varphi_\alpha}{\partial z^i \partial \bar{z}^j} \right)_{i,j} \right\}_{\alpha \in A}$$

Such g' is called

in the Kähler class of g.

The set of Kähler classes

$$\mathcal{C}_V := \{ \text{Kähler metrics on } V \} / \sim$$

$g \sim g'$ iff $\exists\, f : V \to \mathbb{R}$ s.t.

$$g_\alpha' = g_\alpha + \left(\frac{\partial^2 f \circ \varphi_\alpha}{\partial z^i \partial \bar{z}^j} \right)_{i,j}$$

is an open convex cone of

a finite dimensional vector space $H^{1,1}(X, \mathbb{R})$

$$\cap$$

$$H^2(X, \mathbb{R})$$

<u>Rem</u> $g \sim g'$ iff $[\omega] = [\omega'] \in H^2(X, \mathbb{R})$.

§ 3. K-stability

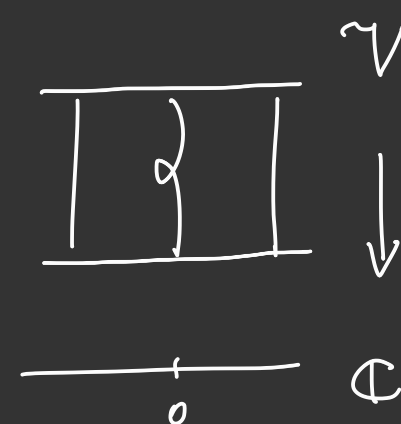
Degeneration of variety

V : a variety of dim n.

A <u>test configuration</u> or <u>$\mathbb{C}^\times$-equivariant degeneration</u> is a $\mathbb{C}^\times$-equivariant proper holomorphic map

$$\begin{array}{ccc} \pi : \mathcal{V} & \to & \mathbb{C} \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{C}^\times & & \mathbb{C}^\times \end{array}$$

of varieties endowed w/ a biholomorphism $i : V \xrightarrow{\sim} \pi^{-1}(1)$.



e.g. $V = \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$

$$(X : Y) \mapsto (X^2 : Y^2 \;\; XY)$$

The image is $Z_0 Z_1 - Z_2^2 = 0$

$\mathbb{CP}^2 \circlearrowleft \mathbb{C}^\times \qquad (Z_0 \cdot Z_1 : Z_2) \cdot t = (t Z_0 : Z_1 : Z_2)$

Put

$$\tilde{V} := \{ ((Z_0 . Z_1 : Z_2), t) \in \mathbb{CP}^2 \times \mathbb{C}$$
$$\downarrow \qquad\qquad \mid Z_0 Z_1 - t Z_2^2 = 0 \}$$
$$\mathbb{C} \ni t$$

$$\tilde{V}_0 = \{ (Z_0 : Z_1 : Z_2) \in \mathbb{CP}^2 \mid Z_0 Z_1 = 0 \}$$

When V is a Fano variety

We take

$\mathcal{L}$: a $\mathbb{C}^\times$-equivariant line bundle on $\tilde{V}$

s.t. $i^*\mathcal{L} \simeq \det TV$

We can compactify $\tilde{V}$ by gluing

$V \times \mathbb{C}$ by the $\mathbb{C}^\times$-equiv map $V \times \mathbb{C}^* \to \tilde{V}$

$(x, t) \mapsto i(x)t^{-1}$

$\rightsquigarrow$ $\left(\begin{array}{l} \overline{\tilde{V}} \to \mathbb{CP}^1 \\ \mathcal{L} \text{ extends to } \overline{\mathcal{L}} \text{ on } \overline{\tilde{V}} \end{array} \right.$

Donaldson-Futaki invariant

$$DF(\mathcal{V}, \mathcal{L}) := -\left(c_1(\bar{\mathcal{V}}/\mathbb{C}P^1) \cdot \bar{\mathcal{L}}^n\right) + \frac{n}{n+1}\left(\bar{\mathcal{L}}^{n+1}\right)$$

A Fano variety is called <u>K-stable</u>

if $DF(\mathcal{V}, \mathcal{L}) > 0$ for $\forall$ $(\mathcal{V}, \mathcal{L})$ nontrivial

Yau-Tian-Donaldson conjecture

<u>Thm</u> (Chen-Donaldson-Sun, Tian 2012)

smooth Fano variety V admits a KE met

iff V is K-stable.

Application: Moduli of KE Fanos

<u>Thm</u> (Odaka, Li-Wang-Xu)

There is an algebraic variety parametrizing all KE Fano mfds

(cf. Blum-Liu-Xu '20)

Idea of proof (analyzing Gromov - Hausdorff limit)

Fix $g_0' \in L$

$$\mathrm{Ric}(g_t) = (1-t) g_0' + t g_t \qquad (*_t)$$

$t=0$ $\quad \mathrm{Ric}(g_0) = g_0'$

$t=1$ $\quad \mathrm{Ric}(g_1) = g_1$

t : open

☆ t . closed under K-stability assumption

$\Rightarrow \{ t \in [0.1] \mid (*)_t \text{ has a solution} \}$

is open & closed . $= [0.1]$

$(*_{t_0})$ $\qquad t \nearrow t_0$

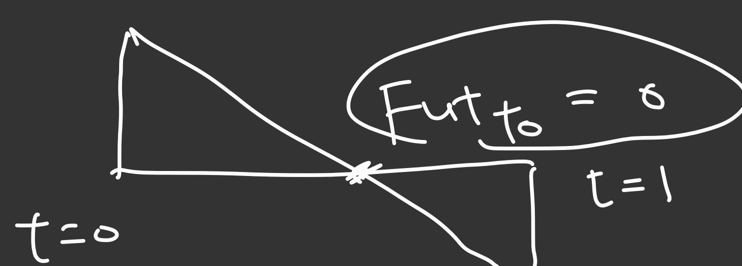
$$(V, g_t) \longrightarrow (V_0, g_{t_0})$$

Gromov-Hausdorff

$$(*_{t_0})$$

$$\Downarrow$$

$$\mathrm{Fut}_{t_0} = 0$$



$\mathrm{Fut}_1 < 0$ $\qquad$ ↯

§ 4 . Kähler - Ricci flow.

Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_t = g_t - Ric(g_t)$$

Kähler-Ricci soliton

$$-L_\xi g = g - Ric(g)$$

Thm (Chen-Sun-Wang '15)

KR flow on smooth Fano V converges to a KR soliton on V_0 in GH topology.

Thm (Dervan - Székelyhidi '16, Han - Li '20)

publish '20

Degeneration $V \rightsquigarrow$ "V_0" minimizes

H-invariant among all degenerations

The minimizers are unique

Thm (Blum - Lin - Xu - Zhuang '21)

The minimizer uniquely exists

even for singular V.

Thm (I '19 + '20)

There is an algebraic variety parametrizing all KRs Fano mfds

$$\rightsquigarrow \quad + \text{ cscK}$$

μ-cscK